Integral Transformations Between Some Function Spaces On Time Scales

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Abstract. In this paper we defined some function spaces on time scale which are Banach spaces respect to supremum norm. We study integral transformations which are carry to some important properties between mentioned above function spaces.

1. Introduction

The calculus on time scales has been introduced by Aulbach and Hilger [**1, 2**] in order to unify discrete and continuous analysis. In [**1, 2, 3**] the concept of integral on time scales is defined by means of an antiderivative(or pre-antiderivative)of function is called Cauchy integral. In [**4**] the Darboux and in [**5, 6, 7**]the Riemann definitions of the integral on time scales are introduced and main theorems of the integral calculus are established. In [**8**] the improper Riemann-Δ Integral is defined which are important in the study of dynamic systems on infinite intervals and properties improper Riemann-Δ integral are established.

Firstly we can give some basic definitions and theorems about the theory of time sclales and Riemann-Δ integration. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The time scale $\mathbb{T}$ is a complete metric space with the usual metric. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standart topology.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$, then t is called *right-dense, and if* $\rho(t) = t$,

then t is called *left-dense*. Points that are right-dense and left-dense are called *dense*. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

For $a, b \in \mathbb{T}$ with $a \leq b$ we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly. (see [**3**])

Let $a < b$ be points in $\mathbb{T}$ and $[a, b]$ the closed interval in $\mathbb{T}$. A *partition* of $[a, b]$ is any finite ordered subset

$$P = \{t_0, t_1, ..., t_n\} \subset [a, b] \qquad \text{where} \qquad a = t_0 < t_1 < ... < t_n = b.$$

We denote the set of all partitions of $[a, b]$ by $\mathcal{P} = \mathcal{P}(a, b)$.

Lemma 1. [**7**]*For every $\delta > 0$ there exists a partition $P = \{t_0, t_1, ..., t_n\} \in \mathcal{P}(a, b)$ such that for each $i \in \{1, 2, ..., n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$.*

Definition 1. [**7**]*We denote by $\mathcal{P}_\delta = \mathcal{P}_\delta(a, b)$ the set of all $P \in \mathcal{P}(a, b)$ that possess the property indicated in Lemma 1.*

Definition 2. [**7**]*Let f be a function on $[a, b]$ and let $P = \{t_0, t_1, ..., t_n\} \in \mathcal{P}(a, b)$. In each interval $[t_{i-1}, t_i)$, where $1 \leq i \leq n$, choose an arbitrary point ξ_i and form the sum*

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

We call S a Riemann sum of f corresponding to $P \in \mathcal{P}$. We say that f is Riemann integrable on $[a, b]$ provided there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann sum S of f corresponding to a partition $P \in \mathcal{P}_\delta$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i), 1 \leq i \leq n$. The number I is called the Riemann Δ-integral of f on $[a, b]$ and we write $\int_a^b f(t)\Delta t = I$.

Theorem 1. [**3**]*Let $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$. Then f is Riemann Δ-integrable from t to $\sigma(t)$ and*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

Theorem 2. [**3**]*Let $a, b \in \mathbb{T}$. Then we have the following:*

i) If $\mathbb{T} = \mathbb{R}$, then a function f on $[a, b]$ is Riemann Δ-integrable from a to b if and only if f is Riemann integrable on $[a, b]$ in the classical sense, and in this case

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the ordinary Riemann integral.

ii) If $\mathbb{T} = \mathbb{Z}$, then every function f defined on $\mathbb{Z}$ is the Riemann Δ-integrable from a to b and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & a < b \\ 0 & a = b \\ \sum_{t=b}^{a-1} f(t) & a > b. \end{cases}$$

Theorem 3. [**7**]*Let f and g integrable functions on $[a,b]$ and let $\alpha \in \mathbb{R}$. Then*

i) αf is integrable and $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

ii) $f+g$ is integrable and $\int_a^b (f+g)(t)\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

iii) fg is integrable.

Theorem 4. [**7**]*Let f be a function defined on $[a,b]$ and let $c \in \mathbb{T}$ with $a < c < b$. If f is integrable from a to c and c to b, then f is integrable from a to b and $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.*

Theorem 5. [**7**]*If f and g are integrable on $[a,b]$ and $f(t) \leq g(t)$ for all $[a,b)$, then $\int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t$.*

Theorem 6. [**7**]*If f is integrable on $[a,b]$ then so is $|f|$ and*

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b |f(t)|\,\Delta t.$$

Now, we assume that $\mathbb{T}$ is unbounded above and $a \in \mathbb{T}$. Let us suppose that

$$\{t_k : k = 0, 1, ...\} \subset \mathbb{T} \quad \text{where} \quad a = t_0 < t_1 < ... \text{ and } \lim_{k \to \infty} t_k = \infty$$

and the function $f : [a, \infty) = \{t \in \mathbb{T} : t \geq a\} \to \mathbb{R}$ is Riemann Δ-integrable from a to any point $A \in \mathbb{T}$ with $A \geq a$. If the integral

$$F(A) = \int_a^A f(t)\Delta t$$

approaches a finite limit as $A \to \infty$, we call that limit the *improper integral of first kind* of f from a to ∞ and we write

(1.1)
$$\int_a^\infty f(t)\Delta t = \lim_{A \to \infty} \int_a^A f(t)\Delta t$$

In such a case we say that the improper integral (1.1) exists or that it is convergent. (see [**3, 8**]).

2. Some Function Spaces and Integral Transformations

Throughout the study we assume that all time scales are unbounded above. Let $\mathbb{T}$ be a such time scale and $[\beta, \infty) \subset \mathbb{T}$. We denote the set of all real valued functions defined on $[\beta, \infty)$ which are Riemann Δ-integrable on every bounded subintervals of $[\beta, \infty)$ by $\mathcal{R}_\mathbb{T}[\beta, \infty)$. Function spaces $C_\mathbb{T}[\beta, \infty)$ and $C_\mathbb{T}^0[\beta, \infty)$ be as follows.

$$C_\mathbb{T}[\beta, \infty) = \left\{ f \in \mathcal{R}_\mathbb{T}[\beta, \infty) : \lim_{t \to \infty} f(t) \text{ exist} \right\}$$

$$C_\mathbb{T}^0[\beta, \infty) = \left\{ f \in \mathcal{R}_\mathbb{T}[\beta, \infty) : \lim_{t \to \infty} f(t) = 0 \right\}.$$

It is evident that $C_\mathbb{T}[\beta, \infty)$ and $C_\mathbb{T}^0[\beta, \infty)$ are Banach spaces with respect to the norm

$$\|f\| = \sup_{a \leq t < \infty} |f(t)|.$$

Note that, if $\mathbb{T} = [\beta, \infty) = \mathbb{N}$ then these function spaces become the space c of convergent sequences and c_0 of null sequences respectively.

We consider time scales $\mathbb{T}_1$ and $\mathbb{T}_2$. Let $[\alpha, \infty) \subset \mathbb{T}_1$ and $[\beta, \infty) \subset \mathbb{T}_2$. We assume that $f \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ and function $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ is Riemann

Δ-integrable with respect to the variable t on every bounded subinterval $[\beta, \infty)$ for each $x \in [\alpha, \infty)$ i.e., $K(x, \circ) \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ for each $x \in [\alpha, \infty)$. If integral

$$(2.1) \qquad (Lf)(x) = \int_{\beta}^{\infty} K(x,t)f(t)\Delta t$$

exists for all $x \in [\alpha, \infty)$ then we transform f to $Lf : [\alpha, \infty) \to \mathbb{R}$. We use notation (X, Y) for all bounded-linear operators from X to Y.

Theorem 7. *Let $[\alpha, \infty) \subset \mathbb{T}_1$, $[\beta, \infty) \subset \mathbb{T}_2$ and $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ be a function such that $K(x, \circ) \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ for each $x \in [\alpha, \infty)$. Suppose the following conditions are satisfied:*

i) $\displaystyle\lim_{x \to x_0} \int_{\beta}^{\infty} |K(x,t) - K(x_0,t)| \, \Delta t = 0 \quad , \quad \forall x_0 \in [\alpha, \infty)$

ii) $\displaystyle M = \sup_{\alpha \leq x < \infty} \int_{\beta}^{\infty} |K(x,t)| \, \Delta t < \infty$

iii) $\displaystyle\lim_{x \to \infty} \int_{\beta}^{y} |K(x,t)| \, \Delta t = 0 \quad , \quad \forall y \in [\beta, \infty).$

Then $L \in (C_{\mathbb{T}_2}^0[\beta, \infty), C_{\mathbb{T}_1}^0[\alpha, \infty))$. Moreover $\|L\| = M$.

PROOF. Let $f \in C_{\mathbb{T}_2}^0[\beta, \infty)$ and we can assume that $\|f\| \neq 0$. By inequality

$$|(Lf)(x) - (Lf)(x_0)| \leq \|f\| \int_{\beta}^{\infty} |K(x,t) - K(x_0,t)| \, \Delta t$$

and condition (i), function Lf is countinous for all $x_0 \in [\alpha, \infty)$. In order to show that $\displaystyle\lim_{x \to \infty} (Lf)(x) = 0$, consider equality

$$(2.2) \qquad (Lf)(x) = \int_{\beta}^{y} K(x,t)f(t)\Delta t + \int_{y}^{\infty} K(x,t)f(t)\Delta t.$$

Since $\displaystyle\lim_{t \to \infty} f(t) = 0$, we can choose real number y such that

$$|f(t)| < \frac{\varepsilon}{2M}$$

for all $t \geq y$ and we have

$$\begin{aligned} \left| \int_{y}^{\infty} K(x,t)f(t)\Delta t \right| &\leq \int_{y}^{\infty} |K(x,t)f(t)| \, \Delta t \\ &< \frac{\varepsilon}{2M} \int_{y}^{\infty} |K(x,t)| \, \Delta t \\ (2.3) \qquad &< \frac{\varepsilon}{2}. \end{aligned}$$

Because of (iii) there exists x_0 such that

$$\int_{\beta}^{y} |K(x,t)| \, \Delta t < \frac{\varepsilon}{2\|f\|}$$

for all $x > x_0$. So we have

$$\left| \int_\beta^y K(x,t) f(t) \Delta t \right| \leq \int_\beta^y |K(x,t) f(t)| \, \Delta t$$

$$< \|f\| \int_\beta^y |K(x,t)| \, \Delta t$$

(2.4) $$< \frac{\varepsilon}{2}$$

for all $x > x_0$. By (2.2),(2.3) and (2.4)we obtain $|(Lf)(x)| < \varepsilon$ for all $x > x_0$. Therefore $Lf \in C_{\mathbb{T}_1}^0[\alpha, \infty)$. Let us show that $\|L\| = M$. Since $\|Lf\| \leq M \|f\|$ for all $f \in C_{\mathbb{T}_2}^0[\beta, \infty)$ we obtain $\|L\| \leq M$. Let arbitrary $\varepsilon > 0$ be given. There exists $x_0 \in [\alpha, \infty)$ such that,

(2.5) $$M - \frac{\varepsilon}{2} < \int_\beta^\infty |K(x_0,t)| \, \Delta t.$$

Because of $\int_\beta^\infty |K(x_0,t)| \Delta t < \infty$ there exist $p \in (\beta, \infty)$ such that,

(2.6) $$\int_p^\infty |K(x_0,t)| \, \Delta t < \frac{\varepsilon}{2}.$$

Consider function $f \in C_{\mathbb{T}_2}^0[\beta, \infty)$ defined by

$$f(t) = \begin{cases} 0 & , \ t > p \\ \mathrm{sgn} K(x_0,t) & , \ t \leq p. \end{cases}$$

It is clear that $\|f\| = 1$. By (2.5) and (2.6) we have

$$\|Lf\| = \sup_{\alpha \leq x < \infty} |(Lf)(x)| \geq |(Lf)(x_0)| = \int_\beta^p |K(x_0,t)| \, \Delta t > M - \varepsilon.$$

So $\|L\| \geq M$. Therefore $\|L\| = M$. □

Theorem 8. *Let $[\alpha, \infty) \subset \mathbb{T}_1$, $[\beta, \infty) \subset \mathbb{T}_2$ and $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ be a function such that $K(x, \circ) \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ for each $x \in [\alpha, \infty)$. Suppose the following conditions are satisfied:*

i) $\lim_{x \to x_0} \int_\beta^\infty |K(x,t) - K(x_0,t)| \, \Delta t = 0 \ , \ \forall x_0 \in [\alpha, \infty)$

ii) $\sup_{\alpha \leq x < \infty} \int_\beta^\infty |K(x,t)| \, \Delta t < \infty$

iii) $\lim_{x \to \infty} \int_\beta^y |K(x,t)| \, \Delta t = 0 \ , \ \forall y \in [\beta, \infty)$

iv) $\lim_{x \to \infty} \int_\beta^\infty K(x,t) \Delta t = 1.$

Then $L \in (C_{\mathbb{T}_2}[\beta, \infty), C_{\mathbb{T}_1}[\alpha, \infty))$. Moreover if $f(t) \to s$ as $t \to \infty$ then $(Lf)(x) \to s$ as $x \to \infty$.

PROOF. For $s = 0$, it is evident by Theorem 7. If $s \neq 0$ repeat by Theorem 7, we have

$$(Lf)(x) = \int_\beta^\infty K(x,t) \left(f(t) - s \right) \Delta t + s \int_\beta^\infty K(x,t) \Delta t.$$

So it is clear that if $f(t) \to s$ as $t \to \infty$ then $(Lf)(x) \to s$ as $x \to \infty$. □

Theorem 9. *Let interval $[\beta, \infty)$ occures isolated points of time scale $\mathbb{T}$. If $F \in C_{\mathbb{T}}^*[\beta, \infty)$ where is dual space of $C_{\mathbb{T}}[\beta, \infty)$. Then there exists real number b and sequence $(b_n) \in l_1$ such that*

$$(2.7) \qquad F(f) = b \lim_{t \to \infty} f(t) + \sum_{n=1}^{\infty} b_n f(t_n)$$

for all $f \in C_{\mathbb{T}}[\beta, \infty)$. Moreover, norm of the functional F is

$$(2.8) \qquad \|F\| = |b| + \sum_{n=1}^{\infty} |b_n| \, .$$

On the contrary, if real number b and sequence $(b_n) \in l_1$ given, left side of equality (2.7) is a member of $C_{\mathbb{T}}^*[\beta, \infty)$. Moreover $C_{\mathbb{T}}^*[\beta, \infty)$ and l_1 are isomorphic spaces.

PROOF. The contrary side is straightforward. Since members of the set $[\beta, \infty)$ are isolated points, we can denote the set $[\beta, \infty)$ by $[\beta, \infty) = \{t_1, t_2, ...\}$ where $\beta = t_1 < t_2 < ...$ and $t_k \to \infty$ as $k \to \infty$. Let $F \in C_{\mathbb{T}}^*[\beta, \infty)$. The set $\{e, e_1, e_2, ...\}$ is a Schauder basis for $C_{\mathbb{T}}[\beta, \infty)$ where $e \equiv 1$ and $e_i(t_j) = \delta_{ij}$ (δ_{ij} is Kronecker delta). For any member f of $C_{\mathbb{T}}[\beta, \infty)$ is expressed by

$$f = le + \sum_{n=1}^{\infty} \left(f(t_n) - l\right) e_n$$

where $l = \lim_{t \to \infty} f(t)$. By linearity and continuity of F we have

$$(2.9) \qquad F(f) = lF(e) + \sum_{n=1}^{\infty} \left(f(t_n) - l\right) F(e_n)$$

for all $f \in C_{\mathbb{T}}[\beta, \infty)$. Consider function $f \in C_{\mathbb{T}}[\beta, \infty)$ defined by

$$f(t_n) = \begin{cases} \operatorname{sgn} F(e_n) & 1 \leq n \leq r \\ 0 & n > r \end{cases}$$

for all $r \geq 1$. Since $\|f\| = 1$ and $|F(f)| \leq \|F\| \|f\|$, we have

$$(2.10) \qquad |F(f)| = \sum_{n=1}^{r} |F(e_n)| \leq \|F\|$$

for all $r \geq 1$. By (2.10) we obtain

$$\sum_{n=1}^{\infty} |F(e_n)| \leq \|F\| < \infty.$$

It means that $\sum_{n=1}^{\infty} F(e_n)$ is absolute convergent. Let $b = F(e) - \sum_{n=1}^{\infty} F(e_n)$ and $b_n = F(e_n)$. By equality (2.9) we get

$$(2.11) \qquad F(f) = bl + \sum_{n=1}^{\infty} b_n f(t_n).$$

Since $|l| \leq \|f\|$ and by (2.11) we have

$$|F(f)| \leq \left(|b| + \sum_{n=1}^{\infty} |b_n| \right) \|f\| \, .$$

Therefore $\|F\| \leq |b| + \sum_{n=1}^{\infty} |b_n|$. Now consider to function $f \in C_{\mathbb{T}}[a, \infty)$ defined by

$$f(t_n) = \left\{ \begin{array}{ll} \mathrm{sgn} b_n & 1 \leq n \leq r \\ \mathrm{sgn} b & n > r. \end{array} \right.$$

It is obvious that $\|f\| = 1$ and $f(t) \to \mathrm{sgn} b$ as $t \to \infty$. By $r \to \infty$ in

$$|F(f)| = \left| |b| + \sum_{n=1}^{r} |b_n| + \sum_{n=r+1}^{\infty} b_n \mathrm{sgn} b \right| \leq \|F\|,$$

we obtain

$$|b| + \sum_{n=1}^{\infty} |b_n| \leq \|F\|.$$

For isomorphism $C_{\mathbb{T}}^*[\beta, \infty)$ to l_1 consider the operator $T : C_{\mathbb{T}}^*[\beta, \infty) \to l_1$ defined by $T(F) = (b, b_1, b_2, ...)$, it is evident that $\|T(F)\| = |b| + |b_1| + |b_2| + ... = \|F\|$ so T preserves norm. $\square$

Theorem 10. *Let interval $[\beta, \infty)$ occures isolated points of time scale $\mathbb{T}_2$ and $[\alpha, \infty)$ be a subinterval of $\mathbb{T}_1$. If $L \in (C_{\mathbb{T}_2}^0[\beta, \infty), C_{\mathbb{T}_1}^0[\alpha, \infty))$ then there exists the function $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ such that $K(x, \circ) \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ for each $x \in [\alpha, \infty)$ which is satisfied equality (2.1). Moreover, has the following properties:*

i) $\|L\| = \sup_{\alpha \leq x < \infty} \int_{\beta}^{\infty} |K(x, t)| \Delta t < \infty$

ii) $\lim_{x \to \infty} \int_{\beta}^{y} |K(x, t)| \Delta t = 0$, $\forall y \in [\beta, \infty).$

PROOF. Let $[\beta, \infty) = \{t_1, t_2, ...\}$ where $\beta = t_1 < t_2 < ...$ and $t_k \to \infty$ as $k \to \infty$. The set $\{e, e_1, e_2, ...\}$ is a Schauder basis for $C_{\mathbb{T}_2}^0[\beta, \infty)$. For any member f of $C_{\mathbb{T}_2}^0[\beta, \infty)$ is expressed by

$$f = \sum_{k=1}^{\infty} f(t_k) e_k.$$

Let $b_k(x) = (Le_k)(x)$ and the function $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ defined by $K(x, t_k)(t_{k+1} - t_k) = b_k(x)$. Since $L \in (C_{\mathbb{T}_2}^0[\beta, \infty), C_{\mathbb{T}_1}^0[\alpha, \infty))$ we have,

$$
\begin{aligned}
(Lf)(x) &= L\left(\sum_{k=1}^{\infty} f(t_k) e_k \right)(x) \\
&= \sum_{k=1}^{\infty} f(t_k)(Le_k)(x) \\
(2.12) \qquad &= \sum_{k=1}^{\infty} f(t_k) b_k(x) \\
&= \sum_{k=1}^{\infty} K(x, t_k) f(t_k)(t_{k+1} - t_k) \\
(2.13) \qquad &= \int_{\beta}^{\infty} K(x, t) f(t) \Delta t.
\end{aligned}
$$

By the hypothesis, we know that $Lf \in C^0_{\mathbb{T}_1}[\alpha, \infty)$ for all $f \in C^0_{\mathbb{T}_2}[\beta, \infty)$. Hence, it is clear that $Le_k \in C^0_{\mathbb{T}_1}[\alpha, \infty)$. This means that the function $K(x, t_k) \to 0$ as $x \to \infty$ for all t_k. Thus we obtain (ii). Let us show that $\|L\| = \sup\limits_{\alpha \leq x < \infty} \int_\beta^\infty |K(x,t)| \Delta t$.

By Theorem 7, it is only need to show this supremum exists. Functionals $L_x : C^0_{\mathbb{T}_2}[\beta, \infty) \to \mathbb{R}$ defined by $L_x(f) = (Lf)(x)$ are linear for each $x \in [\alpha, \infty)$. Since

$$|L_x(f)| = |(Lf)(x)| \leq \|Lf\| \leq \|L\| \, \|f\|$$

L_x are bounded for each $x \in [\alpha, \infty)$. By the uniform boundedness principle, we have

$$\sup_{\alpha \leq x < \infty} \|L_x\| < \infty.$$

Hence, by the Theorem 9 norm of functionals L_x which have form of (2.12) is

$$\|L_x\| = \sum_{k=1}^\infty |b_k(x)| \, .$$

Therefore we obtain,

$$\begin{aligned}
\sum_{k=1}^\infty |b_k(x)| &= \sum_{k=1}^\infty |K(x, t_k)| \, (t_{k+1} - t_k) \\
&= \int_\beta^\infty |K(x,t)| \, \Delta t.
\end{aligned}$$

$\square$

Theorem 11. *Let interval $[\beta, \infty)$ occures isolated points of time scale $\mathbb{T}_2$ and $[\alpha, \infty)$ be a subinterval of $\mathbb{T}_1$. If $L \in (C_{\mathbb{T}_2}[\beta, \infty), C_{\mathbb{T}_1}[\alpha, \infty))$ and $(Lf)(x) \to s$ as $x \to \infty$ whenever $f(t) \to s$ as $t \to \infty$ for all $f \in C_{\mathbb{T}_2}[\beta, \infty)$ then there exists the function $K : [\alpha, \infty) \times [\beta, \infty) \to \mathbb{R}$ such that $K(x, \circ) \in \mathcal{R}_{\mathbb{T}_2}[\beta, \infty)$ for each $x \in [\alpha, \infty)$ which is satisfied equality (2.1). Moreover, K has the following properties:*

i) $\|L\| = \sup\limits_{\alpha \leq x < \infty} \int_\beta^\infty |K(x,t)| \, \Delta t < \infty$

ii) $\lim\limits_{x \to \infty} \int_\beta^y |K(x,t)| \, \Delta t = 0 \quad , \quad \forall y \in [\beta, \infty)$

iii) $\lim\limits_{x \to \infty} \int_\beta^\infty K(x,t) \Delta t = 1.$

Proof. (i) and (ii) are obvious by Theorem 10. For (iii) we can take constant function $f \equiv 1$ in (2.13). $\square$

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